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Natural convection to power-law fluids from two-dimensional or axisymmetric bodies of arbitrary contour

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Abstract—Momentum and heat transfer in power-law-fluid flow over arbitrarily shaped two-dimensional or axisymmetric bodies are examined theoretically. The Merk type of series expansion technique is used for the analysis. The solutions to the governing equations are obtained as universal functions which are independent of the geometry of the problem. With the universal functions obtained, the examples for a vertical flat plate, a horizontal cylinder, a sphere and a vertical cone are given and their results are also compared with the existing results in the literature

1. INTRODUCTION

HEAT TRANSFER in non-Newtonian fluids from external surfaces of bodies of various geometries has been the subject of numerous investigations during the past decades. The interest in this subject still continues.

Acrivos [1] was apparently the first to investigate the natural convection behavior of non-Newtonian fluid flow from a body with an isothermal surface. Since then quite a number of works have been successfully carried out [2–12]. An excellent review on this subject of convective heat transfer in non-Newtonian fluids has recently been made by Shenoy and Mashelkar [13].

Most of the investigations on free convection in non-Newtonian fluids are concerned with a simplified model neglecting the convective term in the governing momentum equation under the assumption of very large Prandtl number and with simple geometries such as a flat plate or a horizontal cylinder.

In this analysis, we propose an exact solution for which the convective term is retained in the governing momentum equation, and is valid for general two-dimensional or axisymmetric bodies. Although, in general, the Prandtl number for a non-Newtonian fluid is large, the effect of the convective term on the heat transfer rate is investigated here.

2. PROBLEM STATEMENT AND MATHEMATICAL FORMULATION

Consideration is given to the steady, natural convective power-law fluid in laminar boundary layer flow over two-dimensional or axisymmetric bodies of uniform surface temperature T_w situated in an infinite ambient fluid of undisturbed temperature T_∞. The flow situation is illustrated in Fig. 1. The coordinate $\bar{x}$ is the distance measured along the surface from the lower stagnation point when the surface is heated, or from the upper stagnation point when the surface is cooled; $\bar{y}$ is the distance along the outer normal to the body. The corresponding velocity components are $\bar{u}$ and $\bar{v}$. For rotationally symmetric bodies, $\bar{r}$, which is a function of $\bar{x}$ only, is the radial distance measured from the axis of symmetry to the surface of the body.

Furthermore, constant properties are postulated, except for the density in the buoyancy term. As usual, the frictional dissipation term in the energy equation is neglected.

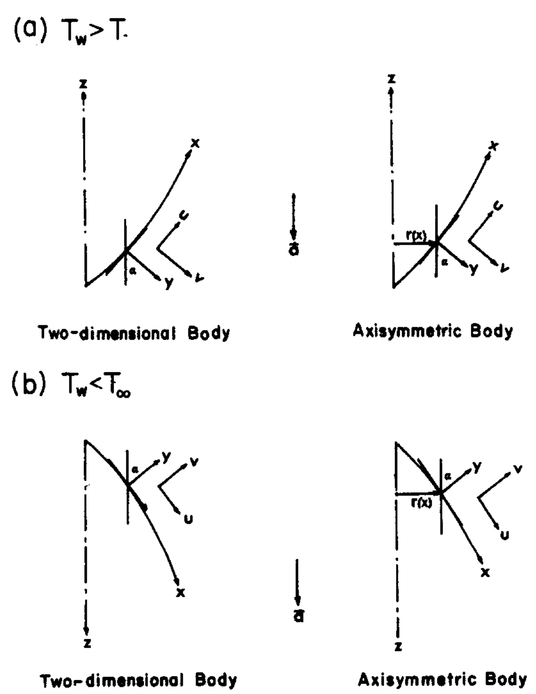

FIG. 1. Physical model and coordinate system.

NOMENCLATURE

a_x acceleration vector in $\bar{x}$-direction
C_p specific heat
E energy factor, equation (22)
f dimensionless stream function, equation (13)
f_1, f_2, f_3 universal stream functions, equation (23)
G_R generalized Grashof number, equation (6)
G_r Grashof number for Newtonian fluids
K consistency index
k thermal conductivity
L reference length
n flow behavior index
Nu Nusselt number
P_R generalized Prandtl number, equation (6)
$\bar{r}$ coordinate measured in the radial direction
T_w surface temperature
T_∞ temperature outside the boundary layer
U velocity function, equation (12)
$\bar{u}$ velocity component in $\bar{x}$-direction
$\bar{v}$ velocity component in $\bar{y}$-direction
$\bar{x}$ streamwise coordinate measured along the surface
$\bar{y}$ coordinate measured along the outer normal to the body.

Greek symbols
α angle between the local gravitational acceleration vector and the outward normal to the body contour
β coefficient of thermal expansion
γ term defined in equation (18)
η transformed dimensionless coordinate, equation (14)
θ dimensionless temperature function, equation (6)
$\theta_1, \theta_2, \theta_3$ universal temperature function, equation (24)
Λ generalized wedge parameter, equation (17)
ξ transformed dimensionless coordinate, equation (14)
ρ fluid density
τ_w shear stress at the wall
ϕ term defined in equation (5)
ψ stream function, equation (13).

The governing boundary layer equations are then

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -a_x\beta(T-T_\infty) + \frac{K}{\rho}\frac{\partial}{\partial \bar{y}}\left[\frac{\partial \bar{u}}{\partial \bar{y}}\left|\frac{\partial \bar{u}}{\partial \bar{y}}\right|^{n-1}\right] \tag{2}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial \bar{y}^2}. \tag{3}$$

The boundary conditions associated with the problem are

$$\begin{aligned} &\bar{u}(\bar{x},0) = \bar{v}(\bar{x},0) = 0, \quad && T(\bar{x},0) = T_w \\ &\bar{u}(\bar{x},\infty) \to 0, && T(\bar{x},\infty) \to T_\infty \end{aligned} \tag{4}$$

where β, k, and T are the thermal expansion coefficient, thermal conductivity, and temperature of the fluid, respectively. The quantities K and n are empirical constants of the power-law model, and a_x is the component of the acceleration vector in the direction of increasing $\bar{x}$. No distinction is made as to the origin of the field force; e.g. gravitational or centrifugal. For the convenience of later discussion, we write

$$a_x = \mp a\phi(\bar{x})\begin{cases} - & \text{for } T_w > T_\infty \\ + & \text{for } T_w < T_\infty \end{cases} \tag{5}$$

where ϕ is a non-dimensional function of $\bar{x}$, and a is a positive constant having the dimensions of acceleration. Consequently, in the gravitational field, if $T_w < T_\infty$, $a_x = g\sin(\alpha)$, α being the angle between the local gravitational acceleration vector and the outward normal to the body contour. In writing equations (2) and (3), it has been assumed that $\beta|(T_w - T_\infty)| \ll 1$ so that the viscous dissipation is ignored.

By introducing the following dimensionless quantities:

$$x = \bar{x}/L,\ y = \frac{\bar{y}}{L}G_R^{1/2(n+1)},\ r = \bar{r}/L,\ \theta = \frac{T-T_\infty}{T_w - T_\infty}$$

$$P_R = \frac{\rho C_p}{k}\left(\frac{K}{\rho}\right)^{2/(n+1)} L^{(1-n)/(1+n)} \times [L\beta a|T_w - T_\infty|]^{3(n+1)/2(n+1)}$$

$$G_R = \frac{\rho^2 L^{2n}}{K^2}[L\beta a|T_w - T_\infty|]^{2-n}$$

$$u = \frac{\bar{u}}{\sqrt{(L\beta a|(T_w - T_\infty)|)}} \qquad v = \frac{\bar{v}G_R^{1/2(n+1)}}{\sqrt{(L\beta a|(T_w - T_\infty)|)}} \tag{6}$$

where L is a characteristic length, P_R the generalized Prandtl number and G_R the generalized Grashof number, the governing equations become

$$\frac{\partial}{\partial x}(r \cdot u)+\frac{\partial}{\partial y}(r \cdot v)=0 \tag{7}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\theta\phi+\frac{\partial}{\partial y}\left[\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right] \tag{8}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=P_R^{-1}\frac{\partial^2\theta}{\partial y^2} \tag{9}$$

with boundary conditions

$$\begin{aligned} &u(x,0)=v(x,0)=0, \quad &\theta(x,0)=1 \\ &u(x,\infty)\to 0, &\theta(x,\infty)\to 0. \end{aligned} \tag{10}$$

If one compares equations (7), (8) and the associated boundary conditions with those for forced flow (see, e.g. ref. [14]), one recognizes their close resemblance. This leads to proposing a hypothetical or 'equivalent' outer stream dimensionless velocity function, $U(x)$, in such a way that

$$\phi = U\frac{\mathrm{d}U}{\mathrm{d}x} \tag{11}$$

so that

$$U=\left(2\int_0^x \phi(x)\,\mathrm{d}x\right)^{1/2}. \tag{12}$$

The continuity equation (7) is automatically satisfied when a dimensionless stream function, ψ, is introduced, i.e.

$$\psi = [(n+1)\xi]^{-1/(n+1)}f(\xi,\eta) \tag{13}$$

and defined by $ru = \partial\psi/\partial y$, $rv = -\partial\psi/\partial x$.

The x, y coordinate system is transformed into a new dimensionless coordinate system by adopting new dimensionless variables

$$\begin{aligned} \xi &= \int_0^x nU^{2n-1}r^{n+1}\,\mathrm{d}x \\ \eta &= Ury[(n+1)\xi]^{-1/(n+1)} \end{aligned} \tag{14}$$

where r is set to 1 for the two-dimensional case. From the above transformations, one finds

$$u = Uf' \tag{15}$$

$$v=-\frac{nr^nU^{2n-1}}{[(n+1)\xi]^{n/(n+1)}} \times\left\{f+(n+1)\xi\frac{\partial f}{\partial \xi}+(\Lambda+\gamma-1)\eta f'\right\} \tag{16}$$

where the prime denotes differentiation with respect to η.

The 'generalized wedge parameter' Λ and γ are defined respectively by

$$\Lambda=\frac{(n+1)\xi}{U}\frac{\mathrm{d}U}{\mathrm{d}\xi}=\frac{(n+1)\phi\xi}{nU^{2n+1}r^{n+1}} \tag{17}$$

$$\gamma=\frac{(n+1)\xi}{r}\frac{\mathrm{d}r}{\mathrm{d}\xi}. \tag{18}$$

The quantity Λ is a function of n and ξ, i.e. x, and can be calculated explicitly if n and U are given. For Newtonian fluids, $n=1$, and Λ will reduce to the wedge parameter defined by Lin and Chao [15].

The boundary layer equations with associated boundary conditions may be reduced to the following system:

$$|f''|^{n-1}f'''+f''f+\Lambda(\theta-f'^2)=(n+1)\xi\frac{\partial(f',f)}{\partial(\xi,\eta)} \tag{19}$$

$$\frac{E}{nP_R}\theta''+f\theta'=(n+1)\xi\frac{\partial(\theta,f)}{\partial(\xi,\eta)} \tag{20}$$

$$\begin{aligned} &f(\xi,0)=f'(\xi,0)=0, \quad &\theta(\xi,0)=1 \\ &f'(\xi,\infty)\to 0, &\theta(\xi,\infty)\to 0 \end{aligned} \tag{21}$$

where $\partial(\theta,f)/\partial(\xi,\eta)$ and $\partial(f',f)/\partial(\xi,\eta)$ denote the Jacobians.

The quantity E defined by

$$E=\left[\frac{n\Lambda}{\phi U}\right]^{(n-1)/(n+1)} \tag{22}$$

is a special parameter only for the non-Newtonian fluid, and it reduces to 1 for a Newtonian fluid. Like Λ, it depends upon the behavior index n, the body contour x, U, and ϕ.

When $n=1$, $E=1$, the equation pair, equations (19) and (20), reduces to that given by Lin and Chao [15].

The solution to equations (19) and (20) may be written in Merk type series form as modified by Kim *et al.* [14] as

$$\begin{aligned} f(\Lambda,\eta,n) &= f_0(\Lambda,\eta,n)+(n+1)\xi\frac{\mathrm{d}\Lambda}{\mathrm{d}\xi}f_1(\Lambda,\eta,n) \\ &\quad +(n+1)^2\xi^2\frac{\mathrm{d}^2\Lambda}{\mathrm{d}\xi^2}f_2(\Lambda,\eta,n) \\ &\quad +\left[(n+1)\xi\frac{\mathrm{d}\Lambda}{\mathrm{d}\xi}\right]^2 f_3(\Lambda,\eta,n)+\cdots \end{aligned} \tag{23}$$

and

$$\begin{aligned} \theta(\Lambda,\eta,n) &= \theta_0(\Lambda,\eta,n)+(n+1)\xi\frac{\mathrm{d}\Lambda}{\mathrm{d}\xi}\theta_1(\Lambda,\eta,n) \\ &\quad +(n+1)^2\xi^2\frac{\mathrm{d}^2\Lambda}{\mathrm{d}\xi^2}\theta_2(\Lambda,\eta,n) \\ &\quad +\left[(n+1)\xi\frac{\mathrm{d}\Lambda}{\mathrm{d}\xi}\right]^2 \theta_3(\Lambda,\eta,n)+\cdots \end{aligned} \tag{24}$$

Upon substituting both f and θ into equations (19) and (20), and collecting terms free of $\mathrm{d}\Lambda/\mathrm{d}\xi$, and then terms common to, $(n+1)\xi\,\mathrm{d}\Lambda/\mathrm{d}\xi$, $(n+1)^2\xi^2\,\mathrm{d}^2\Lambda/\mathrm{d}\xi^2,\ldots,$ etc., a sequence of ordinary differential equations is obtained. The first pair in the sequence (f_0, θ_0) is

$$|f_0''|^{n-1}f_0'''+f_0f_0''+\Lambda(\theta_0-(f_0')^2)=0 \tag{25}$$

and

$$\frac{E}{nP_R}\theta_0'' + f_0\theta_0' = 0 \tag{26}$$

with boundary conditions

$$\begin{aligned} &f_0(\Lambda,0) = f_0'(\Lambda,0) = 0, \quad \theta_0(\Lambda,0) = 1 \\ &f_0'(\Lambda,\infty) \to 0, \quad \theta_0(\Lambda,\infty) \to 0. \end{aligned} \tag{27}$$

The remaining differential equations with their associated boundary conditions are

$$|f_0''|^{n-1} f_1''' + f_0 f_1'' + f_0'' f_1 + \Lambda(\theta_1 - 2f_0' f_1') + (n+1)(f_0'' f_1 - f_0' f_1) = \frac{\partial(f_0', f_0)}{\partial(\Lambda,\eta)} \tag{28}$$

$$\frac{E}{nP_R}\theta_1'' + (f_0\theta_1' + f_1\theta_0') = (n+1)(f_0'\theta_1 - \theta_0' f_1) + \frac{\partial(\theta_0, f_0)}{\partial(\Lambda,\eta)} \tag{29}$$

$$|f_0''|^{n-1} f_2''' + f_0 f_2'' + f_0'' f_2 + (f_0'' f_1 - f_0' f_1') - \Lambda(\theta_2 - 2f_0' f_2') - 2(n+1)(f_0'' f_2 - f_0' f_2') = 0 \tag{30}$$

$$\frac{E}{nP_R}\theta_2'' + (\theta_2' f_0 + \theta_0' f_2) - (f_0'\theta_1 - \theta_0' f_1) + \frac{\partial(\theta_0, f_0)}{\partial(\Lambda,\eta)} = 2(n+1)(f_0\theta_2 - \theta_0' f_2) \tag{31}$$

$$\begin{aligned} &f_i(\Lambda,0) = f_i'(\Lambda,0) = \theta_i(\Lambda,0) = 0 \\ &f_i'(\Lambda,\infty) \to 0, \quad \theta_i(\Lambda,\infty) \to 0 \quad (i = 1,2). \end{aligned} \tag{32}$$

In obtaining equations (25)–(31), the term $|f''|^{n-1}$ has been approximated by $|f_0''|^{n-1}$. This approximation will be verified later by the fact that the first term of series (23) dominates the whole series.

All equation sets f_i and θ_i $(i = 0, 1, 2)$ can be regarded as ordinary differential equations. It should be noted that f_i and θ_i are universal in the sense that, for a given set of Λ, E, P_R, and n, they may be evaluated once and for all. In fact the quantity E/nP_R appearing in equation (31) may be treated as a single parameter, but for the convenience of comparison and discussion, the parameters E, P_R, and n are treated separately. These universal functions have been evaluated numerically by using the fourth-order Runge–Kutta method, and are tabulated for a range of Prandtl numbers which are generally adequate for technological applications [16]. Parts of the tabulated functions are reproduced in Tables 1–3. Although no detailed error analysis has been attempted, the values presented are believed to be accurate to within four to six significant digits. The best way to check the accuracy is to compare them with published data.

3. COMPARISON AND DISCUSSION

To demonstrate the capabilities of the present method of analysis, the local heat transfer for isothermal objects of several geometrical configurations have been examined. A few selected instances are also presented. The generalized shear stress at the wall for power-law fluids can be written in a form which is

Table 1. Wall derivatives of universal functions ($n = 0.50$, $P_R = 100$)

E	Λ	f_0''	θ_0'	$f_1'' \times 10$	$\theta_1' \times 10$	$f_2'' \times 10$	$\theta_2' \times 10^2$
0.20	0.20	0.00858303	−0.59876389	−0.01112	−1.99150	0.0247	4.5845
	0.40	0.01986454	−0.79381729	−0.01287	−1.31116	0.0285	3.0169
	0.60	0.03241876	−0.93553297	−0.01400	−1.02532	0.0310	2.3584
	0.80	0.04585670	−1.05068080	−0.01480	−0.85766	0.0328	1.9709
	1.00	0.05997438	−1.14923810	−0.01573	−0.75421	0.0348	1.7380
	1.20	0.07464249	−1.23619480	−0.01603	−0.67146	0.0354	1.5428
	1.40	0.08977202	−1.31447630	−0.01672	−0.61622	0.0370	1.4195
	1.60	0.10529784	−1.38596880	−0.01697	−0.56218	0.0375	1.2909
0.60	0.20	0.01328172	−0.47977373	−0.01899	−1.70044	0.0419	4.0164
	0.40	0.03069350	−0.63546643	−0.02042	−1.12216	0.0451	2.4106
	0.60	0.04996748	−0.74773425	−0.02184	−0.80751	0.0475	1.8707
	0.80	0.07052022	−0.83856551	−0.02296	−0.67287	0.0499	1.5553
	1.00	0.09201670	−0.91587252	−0.02395	−0.58701	0.0520	1.3556
	1.20	0.11426286	−0.98375792	−0.02430	−0.51807	0.0538	1.1901
	1.40	0.13712662	−1.04461870	−0.02553	−0.47997	0.0555	1.1087
	1.60	0.16051752	−1.10001940	−0.02588	−0.43811	0.0564	1.0070
1.00	0.20	0.01621356	−0.43182252	−0.02139	−1.41749	0.0474	3.2610
	0.40	0.03742822	−0.57156355	−0.02479	−0.92431	0.0552	2.1146
	0.60	0.06087366	−0.67213931	−0.02686	−0.71796	0.0608	1.6236
	0.80	0.08578700	−0.75308825	−0.02814	−0.59744	0.0639	1.3436
	1.00	0.11178444	−0.82180064	−0.02895	−0.51622	0.0654	1.1594
	1.20	0.13863422	−0.88200235	−0.02969	−0.46052	0.0666	1.0370
	1.40	0.16618309	−0.93588037	−0.03027	−0.41750	0.0672	0.9429
	1.60	0.19432735	−0.98485785	−0.03060	−0.38587	0.0672	0.8760

Table 2. Wall derivatives of universal functions ($n = 0.70$, $P_R = 100$)

E	Λ	f''_0	θ'_0	$f''_1\times 10$	$\theta'_1\times 10$	$f''_2\times 10$	$\theta'_2\times 10^2$
0.20	0.20	0.02596181	−1.03716210	−0.03541	−2.15682	0.0718	4.6319
	0.40	0.05075480	−1.29669380	−0.03454	−1.34075	0.0703	2.8662
	0.60	0.07503995	−1.47653810	−0.03380	−1.01710	0.0689	2.1675
	0.80	0.09896967	−1.61838360	−0.03325	−0.83457	0.0679	1.7729
	1.00	0.12262109	−1.73724780	−0.03259	−0.72006	0.0663	1.5314
	1.20	0.14604213	−1.84046420	−0.03236	−0.63284	0.0658	1.3442
	1.40	0.16926565	−1.93222570	−0.03178	−0.57179	0.0646	1.2141
	1.60	0.19231537	−2.01517690	−0.03161	−0.52093	0.0644	1.1034
0.60	0.20	0.03670394	−0.80486003	−0.05143	−1.63252	0.1012	3.5686
	0.40	0.07159650	−1.00476550	−0.04968	−1.10551	0.1063	2.0851
	0.60	0.10566498	−1.14274700	−0.04815	−0.77206	0.1013	1.6012
	0.80	0.13914935	−1.25124710	−0.04710	−0.63509	0.0973	1.3314
	1.00	0.17217232	−1.34193710	−0.04630	−0.54640	0.0945	1.1545
	1.20	0.20480901	−1.42050410	−0.04539	−0.48093	0.0920	1.0173
	1.40	0.23711152	−1.49020030	−0.04444	−0.43326	0.0895	0.9186
	1.60	0.26911686	−1.55307280	−0.04401	−0.39602	0.0884	0.8408
1.00	0.20	0.04304010	−0.71447656	−0.06138	−1.42527	0.1195	3.1339
	0.40	0.08383433	−0.89105783	−0.05893	−0.88789	0.1161	1.9279
	0.60	0.12358921	−1.01266540	−0.05701	−0.67378	0.1128	1.4537
	0.80	0.16260109	−1.10811110	−0.05542	−0.55460	0.1096	1.1934
	1.00	0.20102090	−1.18775180	−0.05413	−0.47450	0.1074	1.0152
	1.20	0.23894219	−1.25663690	−0.05310	−0.42023	0.1051	1.8992
	1.40	0.27643028	−1.31765430	−0.05218	−0.37802	0.1034	0.8071
	1.60	0.31353255	−1.37262360	−0.05138	−0.34576	0.1016	0.7384

analogous to that for Newtonian fluids as

$$\frac{\tau_w}{G_R^{n/2(n+1)}\left[\dfrac{\beta a|T_w-T_\infty|}{L}\right]^{n/2}K} = \frac{(U^2r)^n}{[(n+1)\xi]^{n/(n+1)}}|f''(\Lambda,0,n)|^n \quad (33)$$

where

$$f''(\Lambda,0,n) = f''_0(\Lambda,0,n)+(n+1)\xi\frac{d\Lambda}{d\xi}f''_1(\Lambda,0,n) + (n+1)^2\xi^2\frac{d^2\Lambda}{d\xi^2}f''_2(\Lambda,0,n)+\cdots \quad (34)$$

In a similar way, the local heat flux is defined in terms

Table 3. Wall derivatives of universal functions ($P_R = 100$)

n	E	Λ	f''_0	θ'_0	$f''_1\times 10$	$\theta'_1\times 10$	$f''_2\times 10$	$\theta'_2\times 10^2$
0.50	0.34326	1.49619	0.12028242	−1.20918220	−0.02073	−0.51934	0.0452	1.1193
	0.54292	1.48473	0.14186399	−1.09347250	−0.02399	−0.46331	0.0519	1.0635
	0.84866	1.43858	0.16167984	−0.98116448	−0.02871	−0.42873	0.0617	0.9885
	1.08385	1.36034	0.16588548	−0.91069683	−0.03098	−0.41458	0.0682	0.9544
	1.40056	1.09039	0.14265973	−0.79375038	−0.03332	−0.44851	0.0706	1.0391
	1.53577	0.66698	0.08175254	−0.64203109	−0.03223	−0.60260	0.0875	1.4119
	1.53944	0.36932	0.04052499	−0.50872100	−0.02970	−0.87864	0.0610	2.0680
0.67	0.51941	1.24342	0.18922056	−1.42168320	−0.03844	−0.48537	0.0777	1.0326
	0.68144	1.23487	0.20401151	−1.33074800	−0.04489	−0.45691	0.0931	0.9694
	1.02496	1.13409	0.21498401	−1.17772450	−0.05250	−0.44830	0.1058	0.9807
	1.12215	1.05459	0.20475480	−1.12410160	−0.05442	−0.45684	0.1104	0.9924
	1.18910	0.93737	0.18581948	−1.06790140	−0.05071	−0.48662	0.0985	1.0624
	1.23052	0.78365	0.15765130	−1.00040080	−0.05531	−0.54067	0.1112	1.1726
	1.24901	0.58524	0.11898888	−0.90773786	−0.05763	−0.66099	0.1182	1.4431
	1.24621	0.33007	0.06768436	−0.75454546	−0.05735	−0.97160	0.1156	2.1140
0.83	0.72872	1.10010	0.25762999	−0.15597539	−0.00612	−0.04477	0.0115	0.0916
	0.82789	1.09320	0.26516929	−0.15079558	−0.00646	−0.04347	0.0221	0.0892
	0.93718	1.06509	0.26812893	−0.14511447	−0.00664	−0.04267	0.0124	0.0875
	1.00242	1.01663	0.26246141	−0.14085415	−0.00710	−0.04345	0.0132	0.0910
	1.04546	0.94520	0.24958995	−0.13659754	−0.00697	−0.04521	0.0128	0.0940
	1.07356	0.84667	0.22904496	−0.13162523	−0.00709	−0.04922	0.0132	0.1015
	1.08998	0.71483	0.19930033	−0.12511812	−0.00776	−0.05402	0.0146	0.1114
	1.09641	0.54041	0.15747351	−0.11557619	−0.00834	−0.06568	0.0153	0.1394

Table 4. Comparison oi generalized local Nusselt number for vertical flat plate ($n = 0.5$)

X	Acrivos [1]†	Shenoy and Ulbrecht [6]†	Chen [3]†	Present ($P_R = 100$)
	$Nu_x/G_R^{1/(2(n+1))}P_R^{n/(3n+1)}$			
0.0033	1.5802	1.49431	1.4273	1.4428
0.0703	0.8571	0.81056	0.7742	0.7787
0.2100	0.6886	0.65120	0.6220	0.6243
0.5290	0.5725	0.5413	0.5171	0.5179
0.8009	0.5268	0.4982	0.4759	0.4761
1.1796	0.4875	0.4611	0.4404	0.4404
1.6972	0.4533	0.4287	0.4095	0.4091
2.3915	0.4233	0.4003	0.3824	0.3816
3.2979	0.3967	0.3754	0.3583	0.3575
4.5000	0.3731	0.3528	0.3370	0.3358

† Data are for $Pr \gg 1$.

of the generalized local Nusselt number

$$\frac{Nu_x}{G_R^{1/2(n+1)}} = \frac{Ur}{[(n+1)\xi]^{1/(n+1)}}[-\theta'(\Lambda,0,n)] \quad (35)$$

where

$$\theta'(\Lambda,0,n) = \theta'_0(\Lambda,0,n)+(n+1)\xi\frac{d\Lambda}{d\xi}\theta'_1(\Lambda,0,n) + (n+1)^2\xi^2\frac{d^2\Lambda}{d\xi^2}\theta'_2(\Lambda,0,n)+\cdots \quad (36)$$

Once n and the geometry of the object are given, all the necessary information can be obtained. The shear stress at the wall and the local heat transfer can be readily obtained by using the tabulated data of the universal functions.

3.1. *Natural convection over a vertical flat plate*

Considerable attention has been given to the case of non-Newtonian natural convection over a vertical flat plate. The results for the local heat transfer coefficient for $n = 0.5$ are shown in Table 4, along with the data obtained by using the equation presented by Acrivos [1], Shenoy and Ulbrecht [6], and Chen and Wollersheim [8].

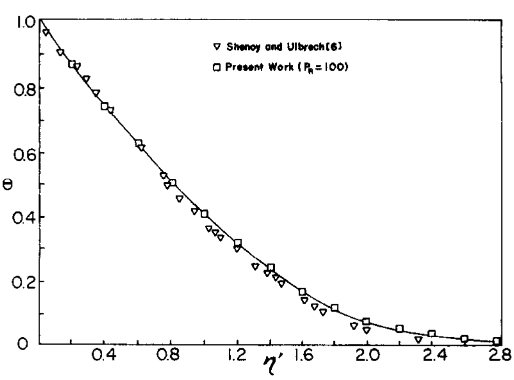

FIG. 2. Variation of θ with η' for 1000 ppm CMC ($n = 0.927$) for a vertical flat plate (η' is defined by equation (3) [6]).

The local Nusselt number computed by the present method is about 10% below that of Acrivos. Since Reilly *et al.* [2] reported that the predictions of Acrivos were about 5–10% higher than the experimental findings, the behavior of the present results is close to the observation of Reilly *et al.*

A comparison of the dimensionless temperature distribution of a flat plate predicted by the present analysis and the experimental findings of Shenoy and Ulbrecht [6] is given in Fig. 2. The agreement is considered to be very good.

Table 5 provides a comparison of average Nusselt numbers (with $P_R = 100$) obtained by the present method, the exact solution of Acrivos, the exact solution of Chen [3], the approximate solution of Tien, and the approximate solution of Shenoy and Ulbrecht [6]. As one can see for $n = 1$, all the values are very close. For $n = 0.5$, our value is exactly the same as that of Chen's [11], but lower than the value obtained by others.

3.2. *Natural convection over a horizontal cylinder*

Consider a long horizontal circular cylinder placed in a power-law fluid. For such a case, the radius of the cylinder is chosen for the reference length L and $\phi = \sin(x)$. It follows that

$$U = [2(1-\cos(x))]^{1/2} \quad (37)$$

$$\xi = \int_0^x n[2(1-\cos(x))]^{(2n-1)/2}\,dx \quad (38)$$

$$\Lambda = \frac{(n+1)\sin(x)\xi}{[2(1-\cos(x))]^{(2n+1)/2}}. \quad (39)$$

The relations among the angle measured from the forward stagnation point and E, ξ, $(n+1)\xi\, d\Lambda/d\xi$, and $(n+1)^2\xi^2\, d^2\Lambda/d\xi^2$ for $n = 0.5$, 0.67, and 0.83 are plotted in Figs. 3 and 4. Using these figures, the shear stress at the wall and the generalized local Nusselt number can be computed.

The data obtained using the present method was compared with the experimental data of Gentry and Wollersheim [7]. The generalized local Nusselt num-

Table 5. Comparison of average Nusselt number for a flat plate

n	Present ($P_R = 100$)	Chen [3]†	Acrivos [1]†	Shenoy and Mashelkar [5]†	Tien [4]†
	$\overline{Nu}/G_R^{1/(2(n+1))}P_R^{n/(3n+1)}$				
0.5	0.569	0.569	0.63	0.5957	0.6098
1.0	0.6723	0.657	0.67	0.6775	0.6838

† Data are valid for $Pr \gg 1$.

bers of five different behavior indices ranging from 0.67 to 1.0 including the Newtonian fluid (i.e. water), are compared. Figure 5 shows the result for water. Figures 6 and 7 are the experimental data of the local free convection results obtained for the 0.053 and 0.055% Carbopol solutions, which agree excellently with the present work. However, the experimental results for the last two solutions, shown in Figs. 8 and 9, are 9–12% higher than that calculated by both the integral-similar method and the present method. No conclusive reasons have been found so far to explain these discrepancies. The influence to the local heat transfer by the flow index decreases with the decrease of flow index. This phenomena is shown in Fig. 10.

Once the information for x, E, Λ, ξ, $(n+1)\xi\,\mathrm{d}\Lambda/\mathrm{d}\xi$, and $(n+1)^2\xi^2\mathrm{d}^2\Lambda/\mathrm{d}\xi^2$ is given, details of the temperature and velocity distribution in the boundary layer can be calculated in a straightforward manner from equations (23) and (24) using either the tabulated universal functions or the computer programs developed in this study.

3.3. *Natural convection from axisymmetric bodies*

The first case considered for the axisymmetric body is the natural convection from a sphere. If the characteristic length is taken equal to the radius of the sphere then

$$r = \sin x \quad \text{and} \quad \phi = \sin x. \tag{40}$$

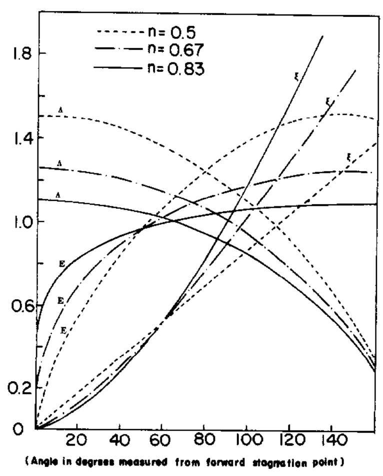

FIG. 3. Variation of Λ, ξ and E along the surface of a horizontal cylinder.

The local Nusselt number can be readily obtained as

$$Nu_x = [2(1-\cos x)]^{1/2}\cdot\sin x \cdot[-\theta'(\Lambda,0,n)]\cdot[(n+1)\xi]^{-1/(n+1)}\cdot G_R^{1/2(n+1)}. \tag{41}$$

Figure 11 shows Nusselt numbers calculated by the present series solution with $P_R = 5500$ as well as by that of Acrivos [1] for $n = 0.927$.

The effect of the convective term on the heat transfer rate can be observed by the increase of P_R. As $P_R = 500$, Nu predicted by the present method is around 4.4% lower than that of Acrivos [1].

For the second case, a vertical cone with its apex points downward. Under this condition, $\phi = \cos\gamma$ and $r = x$. γ is the half angle of the apex and its characteristic length L is set equal to the length of the cone. The following expressions can be easily obtained:

$$\Lambda = \frac{n+1}{4n+3} \tag{42}$$

$$\xi = \frac{2n}{4n+3}(2\cos\gamma)^{(2n-1)/2}(\sin\gamma)^{n+1}X^{(4n+3)/2}. \tag{43}$$

The local Nusselt number is given by

$$Nu_x = [-\theta'(\Lambda,0,n)]\left[\frac{4n+3}{2n(n+1)}\right]^{1/(n+1)} \times(2\cos\gamma)^{-(n-2)/2(n+1)}X^{-n/2(n+1)}\cdot G_R^{1/2(n+1)}. \tag{44}$$

The theoretical predictions on laminar natural convection heat transfer from a vertical cone to a power-law fluid are those of Acrivos [1] and Shenoy [10].

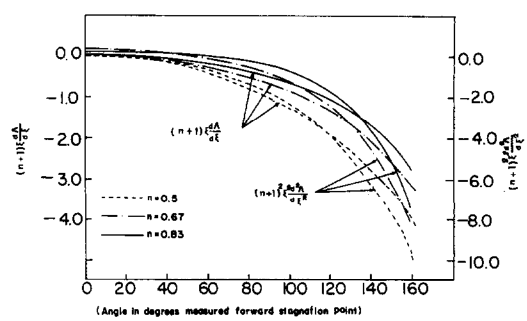

FIG. 4. Variation of $(n+1)\xi\,\mathrm{d}\Lambda/\mathrm{d}\xi$ and $(n+1)^2\xi^2\mathrm{d}^2\Lambda/\mathrm{d}\xi^2$ along the surface of a horizontal cylinder.

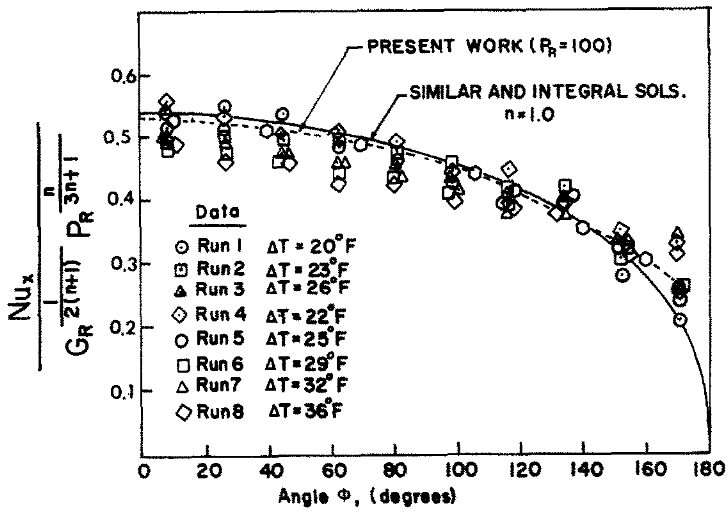

FIG. 5. Comparison of the present work and Gentry and Wollersheim's experimental data [7] (n = 1.0).

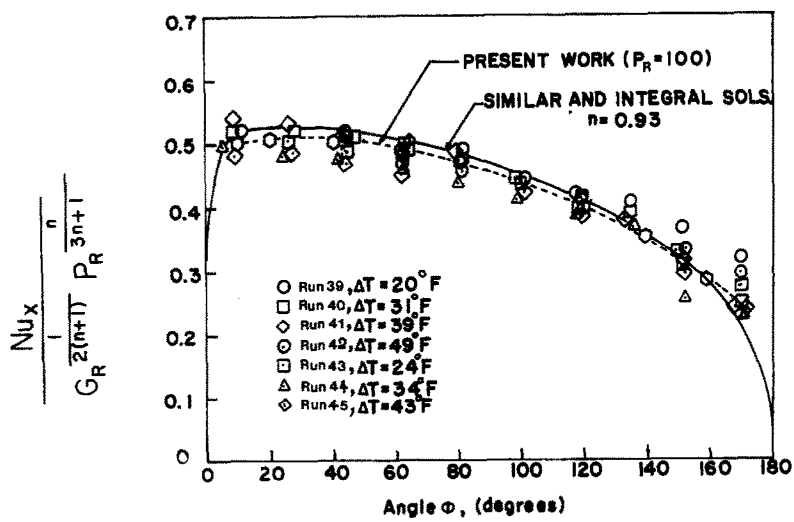

FIG. 6. Comparison of the present work and Gentry and Wollersheim's experimental data [7] (n = 0.93).

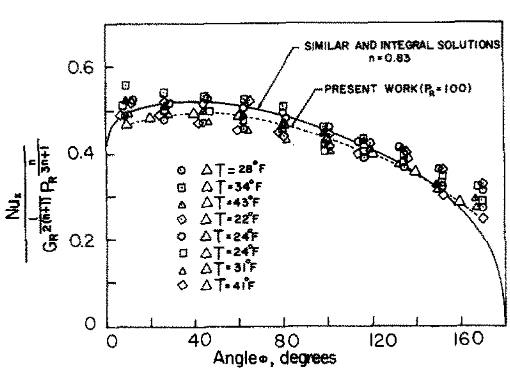

FIG. 7. Comparison of the present work and Gentry and Wollersheim's experimental data [7] (n = 0.83).

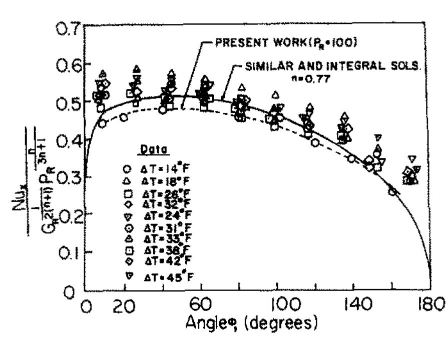

FIG. 8. Comparison of the present work and Gentry and Wollersheim's experimental data [7] (n = 0.77).

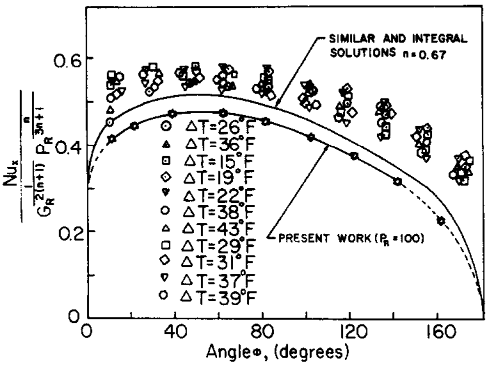

FIG. 9. Comparison of the present work and Gentry and Wollersheim's experimental data [7] (n = 0.67).

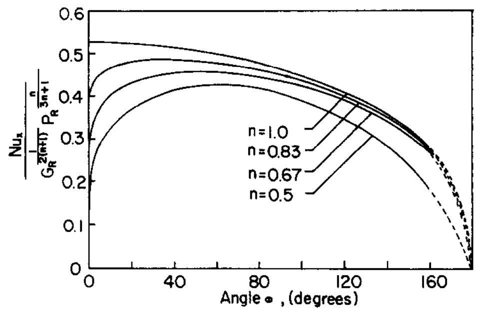

FIG. 10. Effect of flow behavior index on local heat transfer rate over a horizontal cylinder.

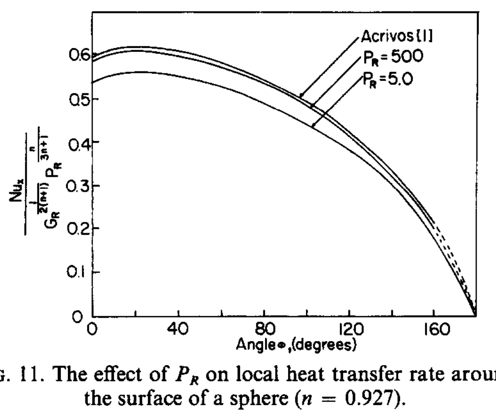

FIG. 11. The effect of P_R on local heat transfer rate around the surface of a sphere (n = 0.927).

Table 6. Comparison of local Nusselt number for a vertical cone ($\gamma = 30°$, $n = 0.927$, $P_R = 500$)

	$Nu_x/G_R^{1/(2(n+1))}P_R^{n/(3n+1)}$		
x	Acrivos [1]†	Shenoy [10]†	Present ($P_R = 500$)
0.06	1.2275	1.1923	1.1699
0.10	1.0830	1.0519	1.0321
0.30	0.8273	0.8036	0.7883
0.50	0.7299	0.7090	0.6955
0.70	0.6721	0.6528	0.6404
1.00	0.6158	0.5982	0.5867
1.50	0.5575	0.5415	0.5312
2.00	0.5196	0.5047	0.4950

† Data are valid for large P_R.

Nusselt numbers calculated by the present method along with those from Acrivos' and Shenoy's methods are given in Table 6 (for n = 0.927, P_R = 500).

It is worthy to note that the present work as well as that of Acrivos does not include the curvature effect. Our results agree with those of Acrivos [1] and Shenoy [10] to within 5%.

4. CONCLUSION

The momentum and heat transfer phenomena occurring in laminar natural convection to non-Newtonian power-law fluids has been theoretically examined. The Merk–Meksyn series expansion method and the generalized coordinate transformation can transform the partial differential momentum and energy equations into two sets of infinite-sequence type ordinary differential equations, respectively. The solutions to these sets of differential equations can be obtained as universal functions which are tabulated once and for all for geometries. The technique presented in this analysis provides a general, accurate, and relatively simple method to analyze the transport phenomena in the laminar boundary layer of power-law fluids. In application, the present results are in good agreement with those obtained from previous experiments or other theoretical work. With the present analysis, the validity of the analysis for large P_r is re-examined. The authors also believe that the results of the velocity and temperature fields obtained by using this analysis can significantly improve the prediction of mass transfer in power-law fluids with heterogeneous surface reaction.

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CONVECTION NATURELLE DE FLUIDES A LOI-PUISSANCE AUTOUR DE CORPS BIDIMENSIONNELS OU AXISYMETRIQUE DE CONTOUR QUELCONQUE

Résumé—On étudie théoriquement le transfert de quantité de mouvement et de chaleur dans un fluide à loi-puissance en écoulement sur des corps de forme quelconque bidimensionnelle ou axisymétrique. La technique de type Merk de développement en série est utilisée pour l'analyse. La solution des équations est obtenue comme des fonctions universelles qui sont indépendantes de la géométrie du problème. Avec les fonctions universelles obtenues, on traite les cas d'une plaque plane verticale, d'un cylindre horizontal, d'une sphère et d'un cône vertical et les résultats sont comparés aux résultats déjà connus.

NATÜRLICHE KONVEKTION BEI FLUIDEN MIT NICHTLINEARER SCHUBSPANNUNG AN ZWEIDIMENSIONALEN ODER ACHSENSYMMETRISCHEN KÖRPERN BELIEBIGER GESTALT

Zusammenfassung—Der Impuls- und Wärmetransport in Fluiden mit nichtlinearem Schubspannungsansatz an beliebig geformten zweidimensionalen oder achsensymmetrischen Körpern wurde theoretisch untersucht. Die Merk'sche Reihenentwicklung wurde für die Untersuchung verwendet. Die Lösung wurde in allgemeiner Form unabhängig von der Geometrie ermittelt. Mit diesen allgemein anwendbaren Gleichungen wurden die senkrechte ebene Platte, der waagerechte Zylinder, die Kugel und ein senkrechter Kegel als Beispiele berechnet und mit den vorhandenen Ergebnissen in der Literatur verglichen.

ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ СТЕПЕННЫХ ЖИДКОСТЕЙ ВБЛИЗИ ДВУМЕРНЫХ ИЛИ ОСЕСИММЕТРИЧНЫХ ТЕЛ ПРОИЗВОЛЬНОЙ ФОРМЫ

Аннотация—Используя технику Мерка разложения в ряды, проведен теоретический анализ переноса импульса и тепла в степенной жидкости, окружающей двумерные или осесимметричные тела произвольной формы. Получено решение основных уравнений в виде универсальных функций, которые не зависят от геометрии задачи. С помощью этих функций исследована естественная конвекция от вертикальной плоской пластины, горизонтального цилиндра, шара и вертикального конуса. Проведено сравнение полученных результатов с опубликованными в литературе данными.